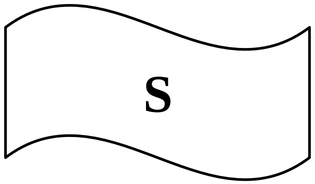
# Lecture- 6

## Stoke’s theorem, Divergence Theorem

Considering a surface **S** having element **dS** and curve **C** denotes the curve :




# Stokes' Theorem

If there is a vector field **A**, then the line integral of **A** taken round C is equal to the surface integral of $\nabla \times A$ taken over **S** :

$$\int_C \mathbf{A} \cdot \mathbf{dr} = \int_S \nabla \times \mathbf{A} \cdot \mathbf{dS} = \int_S \nabla \times \mathbf{A} dS$$

Two-dimensional system

$$\mathbf{A} = A_x \mathbf{i} + A_y \mathbf{j}$$

$$\mathbf{dr} = dx\mathbf{i} + dy\mathbf{j}$$

$$\mathbf{n} dS = dxdy\mathbf{k}$$

$$\nabla \times \mathbf{A} = \left( \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \mathbf{k}$$

$$\int_C (A_x dx + A_y dy) = \iint_S \left( \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) dxdy$$

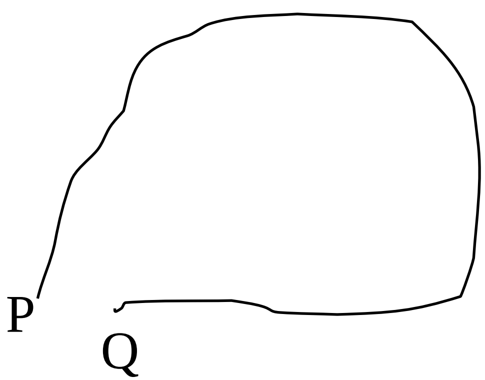
How to make a line to a surface ?

P and Q represent the same point!

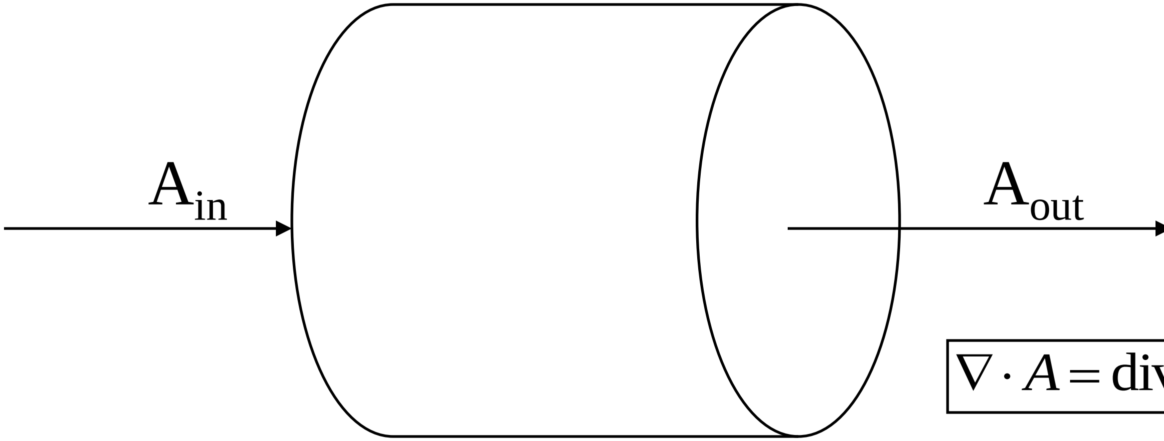
$$\int_P^Q \mathbf{A}\cdot\mathbf{dr} = \int_P^Q (A_x dx + A_y dy + A_z dz)$$

$$\int \mathbf{A}\cdot\mathbf{dS} = \int \mathbf{A}\cdot\mathbf{n}\,dS$$

$$\int_C \mathbf{A}\cdot\mathbf{dr} = \int_S \nabla\times\mathbf{A}\cdot\mathbf{dS}$$



# Gauss' **Divergence** Theorem




$$\nabla \cdot A = \text{div } A$$

The tubular element is "divergent" in the direction of flow.

$$\nabla \cdot \rho \mathbf{u} = \text{div } \rho \mathbf{u}$$

The net rate of mass flow from unit volume

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We also have : The surface integral of the velocity vector **u** gives the net volumetric flow across the surface

$$\int \mathbf{u} \cdot \mathbf{dS} = \int \mathbf{u} \cdot \mathbf{n} dS$$

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The mass flow rate of a closed surface (volume)

$$\int_S \rho \mathbf{u} \cdot \mathbf{dS} = \int_\sigma \nabla \cdot \rho \mathbf{u} d\sigma$$

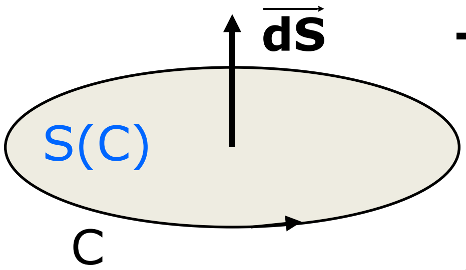
## Stokes' Theorem

$$\int_C \mathbf{A} \cdot \mathbf{dr} = \int_S \nabla \times \mathbf{A} \cdot \mathbf{dS} = \int_S \nabla \times \mathbf{A} dS$$

## Gauss' Divergence Theorem

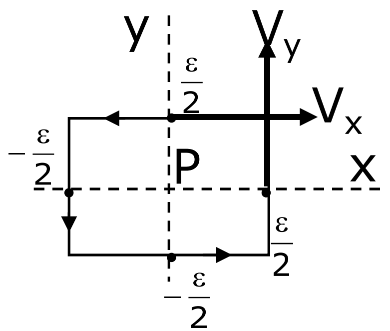
$$\int_S \mathbf{A} \cdot \mathbf{dS} = \int_\sigma \nabla \cdot \mathbf{A} d\sigma$$

- **Stokes formula: vector field global circulation**




**Theorem.** If S(C) is **any** oriented surface delimited by C:

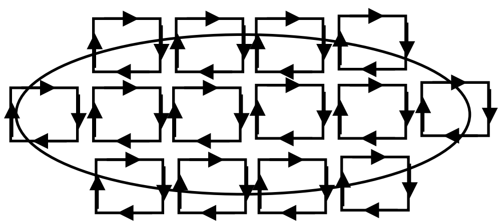
$$\int_C \vec{\mathbf{V}} \cdot \overrightarrow{\mathbf{dr}} = \iint_{S(C)} \mathbf{curl}\ \vec{\mathbf{V}} \cdot \overrightarrow{\mathbf{dS}}$$




**Sketch of proof.**

$$\int_{\varepsilon^2} \vec{\mathbf{V}} \cdot \overrightarrow{\mathbf{dr}} = \left( V(P) + \frac{\varepsilon}{2}\frac{\partial V_y}{\partial x} \right) \cdot \varepsilon + \left( V(P) - \frac{\varepsilon}{2}\frac{\partial V_y}{\partial x} \right) \cdot (-\varepsilon) + O(\varepsilon^3)$$

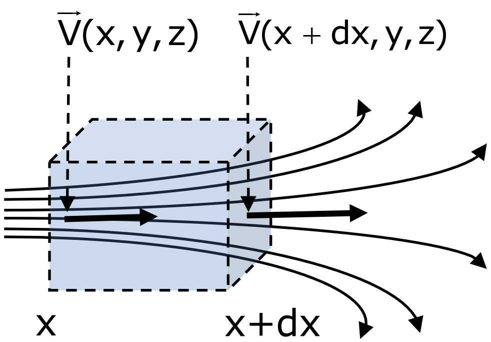
$$+ \left( V(P) + \frac{\varepsilon}{2}\frac{\partial V_x}{\partial y} \right) \cdot (-\varepsilon) + \left( V(P) - \frac{\varepsilon}{2}\frac{\partial V_x}{\partial y} \right) \cdot \varepsilon + O(\varepsilon^3)$$



$$\int_{\varepsilon^2} \vec{\mathbf{V}} \cdot \overrightarrow{\mathbf{dr}} = \left( \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right) \varepsilon^2 + O(\varepsilon^3)$$

… and then extend to **any** surface delimited by C.

- **Divergence Formula: global conservation laws**




**Theorem.** If V(C) is the volume delimited by S

$$\iint_S \vec{\mathbf{V}} \cdot \vec{\mathbf{dS}} = \iiint_{V(S)} \operatorname{div} \vec{\mathbf{V}} \, dV$$

**Sketch of proof**. Flow through the oriented elementary planes x = ctt and x+dx = ctt :

$$-V_x(x,y,z).dydz + V_x(x+dx,y,z).dydz$$

and then extend this expression to the lateral surface of the cube.

Other expression: $V_x(x+dx,y,z).dydz - V_x(x,y,z).dydz = \frac{\partial V_x}{\partial x} dxdydz$

extended to the vol. of the elementary cube: $\left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right) dxdydz$